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1 Multivariate Functions

1.1 Basic definitions:

1.1.1 Domain of a function:

Let D be a set of $n-$tuples of real numbers $(x_1, x_2, ..., x_n)$. A real valued function, $f : D \to \mathcal{R}$ is a rule that assigns a real number, $y = f(x_1, x_2, ..., x_n)$ to each element in D. The set D is the function's **domain**.
eg. Domain of the function, $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ is the entire space of real numbers. Domain of the function, $g(x, y, z) = \frac{1}{x^2+y^2+z^2}$ is all real x, y, z except $(x, y, z) = (0, 0, 0)$.

1.1.2 Range of function:

The set of all possible y values taken on by the function f is called the function's **range**.
eg. Range of f in the above example is $[0, \infty)$. Range of g in the above example is $(0, \infty)$.

Trick question: Which are the independent and dependent variables?

1.1.3 Interior points and Interior of a set:

A point (x_0, y_0) in a region (set) R in the xy plane (equivalently in *space*) is an **interior point** of R if it is the center of a disk that lies entirely in R. The set of all interior points is known as the **interior** of the region, R.

1.1.4 Boundary points and Boundary:

A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points in R. *The boundary point itself need not belong to R.* The set of all boundary points make up the **boundary** of the region.

1.1.5 Open and Closed regions:

A region is **open** if it consists entirely of interior points. eg. $\{(x, y)|x^2 + y^2 < 1\}$. ← Bdy pts are not included in an "open" region.
A region is **closed** if it contains all of its boundary points. eg. $\{(x, y)|x^2 + y^2 \leq 1\}$.

1.1.6 Bounded and Unbounded regions:

A region is **bounded** if it lies inside a disk of fixed radius, else it is **unbounded**.

example: The domain of the function $f(x, y) = \sqrt{y - x^2}$ is *closed* and *unbounded*. The parabola $y = x^2$ is the *boundary* of the domain. The points above the parabola make up the domain's *interior*.

1.1.7 Level curve, graph, surface, level surface:

The set of points where a function f has a constant value $f(x, y) = c$ is called **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface**, $z = f(x, y)$. The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

Sample exercise problems:

1. Find the level curve of the function, $f = \sum_{n=0}^{\infty}\left(\frac{x}{y}\right)^n$ that passes through the point $(1, 2)$.
   Soln. $f(x, y) = \sum_{n=0}^{\infty}(\frac{x}{y})^n$ @$(1, 2)$ $\implies z = \frac{1}{1-\frac{x}{y}} = \frac{y}{y-x}$ @$(1, 2)$ $\implies z = \frac{2}{2-1} = 2 \implies 2 = \frac{y}{y-x} \implies y = 2x$. $\}f(x,y) = y - 2x = 0 = c$
2. Find the level surface of the function, $f(x, y, z) = \sqrt{x - y} - \log z$ @$(3, -1, 1)$.
   Soln. $f(x, y, z) = \sqrt{x - y} - \log z$ @$(3, -1, 1)$ $\implies w = \sqrt{x - y} - \log z$ @$(3, -1, 1)$ $\implies w = \sqrt{3 - (-1)} - \log 1 = 2 \implies \sqrt{x - y} - \log z = 2$.

Difference b/n partial & total derivatives & Geometrical perspective: (A) (B)

(A) Represents the slope of the tangent line to the curve of the f^n in a specific dir^n holding other variables fixed.

1.2 Limits and Continuity

Reading Assignment: Review definitions and properties from page 917-919, textbook.

1.2.1 2-path test for (non) existence of a limit:

If a function $f(x,y)$ has different limits along 2 different paths as $(x,y) \to (x_0,y_0)$, then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

The notion of ***path*** *should be clear in this context. In calculus 1, when limits where introduced, the* ***path*** *almost always was on the real line; here since we are dealing with bi-variate (multivariate) functions,* ***path*** *may imply any curve on the $xy-$plane on which the set of points (x,y) may* `ride` *upon to approach (x_0,y_0). The important thing to know is that only such a curve may be chosen to ensure that the point (x_0,y_0) actually lies on the curve.*

1.2.2 Sample review problem:

1. Show

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & ;(x,y) \neq (0,0), \\ 0 & ;(x,y) = (0,0) \end{cases}$$

is **not** continuous at $(0,0)$.

Soln. : `(Book's method)` Let us choose a path $y = mx$ and analyze the limit at $(0,0)$.

Note $f(x,y)\Big|_{y=mx} = \frac{2xy}{x^2+y^2}\Big|_{y=mx} = \frac{2m}{1+m^2}$. Therefore,

$$\lim_{(x,y)\to(x_0,y_0)\text{along } y=mx} f(x,y) = \frac{2m}{1+m^2}$$

changes with m, and hence according to the 2-path test, $f(x,y)$ is discontinuous at $(0,0)$. □

`(Alternative method)` Let us a choose a sequence $\{(\frac{1}{k},\frac{1}{k})\}$ that will define our path. Clearly, $\{(\frac{1}{k},\frac{1}{k})\} \to (0,0)$ as $k \to \infty$. And since $f(\frac{1}{k},\frac{1}{k}) = \cancel{\frac{1}{2}}$ 1 (typo) for any k, the function sequence $\{f(\frac{1}{k},\frac{1}{k})\} \to \cancel{\frac{1}{2}}$ 1 (typo). Now lets choose a different sequence $\{(\frac{1}{k},0)\} \to (0,0)$ as $k \to \infty$. But $f(\frac{1}{k},0) = 0$ for any k, and so the function sequence $\{f(\frac{1}{k},0)\} \to 0$ as $k \to \infty$. Hence the desired conclusion. □

2 Partial Derivatives

Reading Assignment: Review page 924-929 from your textbook

2.1 Euler's Theorem (Mixed Derivatives):

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a,b) and are **all continuous** at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

Clairaut's th^m

2.1.1 Sample Review Problems

1. Is $f_{xy} = f_{yx}$ always true ? Only when f has continuous partial derivatives!
2. If the limits exist, then is the following always true ? No!

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y)$$

Soln. : `(hint)` Try

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

and use $a = b = 0$.

Total derivative

(I) Consider a bi-variate f^n $f(x,y)$ total derivative w.r.t. x is

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

(II) Implicit f^n $f(x(t), y(t))$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

3 Linearization and Differentials

3.1 Linearization

The *linearization* of a function $f(x,y)$ at a point (x_0,y_0) where f is differentiable is the function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

(B) Represents the slope of the tangent line to the curve of the f^n when all variables are allowed to change.

(III) However, $f(t, x(t), y(t))$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

* $\lim\limits_{x \to 0} \lim\limits_{y \to 0} \dfrac{x^2 - y^2}{x^2 + y^2}$

$= \lim\limits_{x \to 0} \left\{ \lim\limits_{y \to 0} \dfrac{x^2 - y^2}{x^2 + y^2} \right\} = \lim\limits_{x \to 0} 1 = 1$

* $\lim\limits_{y \to 0} \lim\limits_{x \to 0} \dfrac{x^2 - y^2}{x^2 + y^2} = \lim\limits_{y \to 0} -1 = -1$

$\therefore$ lim are NOT interchangeable.