Linear Algebra

Engineering Mathematics In Action

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FIELD (Definition): A field is a set $\mathbb{F}$ of numbers with the property that if $a, b \in \mathbb{F}$, then $a + b$, $a - b$, ab and $\frac{a}{b}$ are also in $\mathbb{F}$ (assuming, of course, that $b \neq 0$ *in the expression* $\frac{a}{b}$.

e.g. $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are fields of numbers

$\mathbb{N}$ and $\mathbb{Z}$ are **not** fields of numbers!

$\mathbb{Q}$ - Rational Numbers
$\mathbb{R}$ - Real Numbers
$\mathbb{C}$ - Complex Numbers

$\mathbb{N}$: Natural Numbers (positive integers)
$\mathbb{Z}$: Integers

VECTOR SPACES (Definition): A vector space, $\boldsymbol{\mathcal{V}}$ consists of a set $\mathbb{V}$ of vectors, a field $\mathbb{F}$ of scalars, and **two** operations:

i. **Vector Addition:** *if* $v, w \in \mathbb{V}$, *then* $v + w \in \mathbb{V}$

ii. **Scalar Multiplication:** $c \in \mathbb{F}$ *and* $v \in \mathbb{V}$ *produces a new vector* $cv \in \mathbb{V}$

These scalars and vectors also satisfy the following **axioms**

i. ***Associativity** of addition:* $(v + u) + w = v + (u + w) \ \ \forall v, u, w \in \mathbb{V}$
ii. ***Associativity** of multiplication:* $(ab)u = a(bu)$, *for any* $a, b \in \mathbb{F}, u \in \mathbb{V}$
iii. ***Distributivity:*** $(a + b)u = au + bu \quad and \quad a(u + v) = au + av$
$\forall\, a, b \in \mathbb{F},\ u \in \mathbb{V},\ v \in \mathbb{V}$
iv. ***Unitarity**:* $1u = u \ \ \forall u \in \mathbb{V}$
v. ***Existence of zero**:* $\exists 0 \in \mathbb{V} \ \ s.t.\ u + 0 = u \quad \forall u \in \mathbb{V}$
vi. ***Negation**:* $For\ every\ u \in \mathbb{V}, \exists (-u) \in \mathbb{V} \quad s.t. \quad u + (-u) = 0 \in \mathbb{V}$

From the notes of Prof. Amrik Sen, Plaksha University

VECTOR SPACES (Examples):

1) Let $\mathbb{V}$ be the set of $n \times 1$ column matrices (vectors), $\mathbb{F}$ be the field of *reals* $\mathbb{R}$, and the laws of vector addition and scalar multiplication are defined as:

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdot \\ \cdot \\ x_n + y_n \end{pmatrix} \quad \text{and} \quad c\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \cdot \\ \cdot \\ cx_n \end{pmatrix}.$$

HW: Verify that the above indeed constitutes a vector space!
(Check that the axioms are satisfied.)

VECTOR SPACES (Examples):

2) Let $\mathbb{V}$ be the set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$, let the field of scalars be $\mathbb{R}$, and let the operations be as usually defined.

HW: Verify that the above indeed constitutes a vector space!

3) Let $\mathbb{V}$ be the set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$ that satisfy the equation $f'' = -f$. (Can you think of any function that satisfies this property? Cosine, Sine?)

Let the field of scalars be $\mathbb{R}$. The operations are defined in the usual manner. *Hint: Suppose* $f_1, f_2 \in \mathbb{V}, c \in \mathbb{R}$; *then* $(f_1 + f_2)'' = f_1'' + f_2'' = -f_1 - f_2 = -(f_1 + f_2)$; *and* $(cf_1)'' = cf_1'' = c(-f_1) = -(cf_1)$.
Are these results consistent with the definition of the vector space?
Also check whether all axioms are compliant?

LINEAR INDEPENDENCE OF VECTORS

Definition (Linearly_dependent_vectors):
Let $\mathcal{V}$ be a vector space and $\mathcal{X} \subset \mathcal{V}$ be a non-empty subset. Then $\mathcal{X}$ is ***linearly dependent*** if there are distinct vectors $v_1, v_2, \ldots, v_k \in \mathcal{X}$, *and scalars* $c_1, c_2, \ldots, c_k$ ***(not all of them zero)****, s.t.* $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$.

This is equivalent to saying that *at least one of the vectors* v_i *can be expressed as a linear combination of the others, i.e.* $v_i = \sum_{j \neq i} -(\frac{c_j}{c_i}) v_j$

Definition (Linearly independent vectors):
A subset which is not linearly dependent is said to be ***linearly independent***. Thus a set of distinct vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if an equation of the form $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$ always implies that $c_1 = c_2 = \ldots = c_k = 0$.

Geometrical Interpretation of Linear Dependence

Let V_1, V_2, V_3 be the vectors in 3D-Euclidean space $\mathbb{R}^3$ with a common origin. If these vectors form a *linearly dependent* set, then one of them, say V_1, can be expressed as a linear combination of the other two: $V_1 = aV_2 + bV_3$. This implies, by the parallelogram law, that the three vectors are **co-planar.**

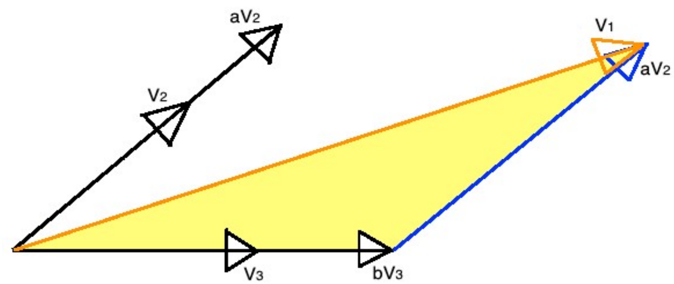


Fig. 1: Linear dependence of vectors is equivalent to coplanar geometry

In fact, **linearly dependent set of vectors with common origin $\Leftrightarrow$ co-planar.**

Can you think of a similar interpretation of vectors in $\mathbb{R}^2$?

Consider what happens when we have three vectors **A**, **B** and **C**, from a common origin, in a 2-dimensional vector space, where –

A = $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ **B** = $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and **C** = $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Can I represent the vector **C** as a linear combination of the vectors **A** and **B** such as **C** = α**A** + β**B** ?

Yes, if we choose α and β as $\alpha = \dfrac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1}$ and $\beta = \dfrac{c_2 a_1 - c_1 a_2}{a_1 b_2 - a_2 b_1}$

Looking at these, we can immediately conclude that this cannot be done if A and B are collinear, because then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

What can you conclude when either *α* or *β* or both become zero?

Example

Show that these vectors are linearly dependent in $\mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

We choose scalars c_1, c_2, c_3 such that -

$$c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives

$$\begin{aligned} -c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 2c_2 - 4c_3 &= 0 \end{aligned}$$

Since the number of unknowns is more than the number of equations, there will be a non-trivial solution

Therefore, the vectors are Linearly Dependent

Example

Are the polynomials $x+1$, $x+2$, x^2-1 linearly independent in the vector space $P_3(\mathbb{R})$?

Notation: $P_3(\mathbb{R})$ is the set of polynomials of less than degree 3 with real coefficients

We choose scalars c_1, c_2, c_3 such that

$$c_1(x+1)+c_2(x+2)+c_3(x^2-1)=0$$

This gives

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 0 \\ c_1 + c_2 \qquad &= 0 \\ c_3 \qquad\qquad &= 0 \end{aligned}$$

Clearly, this can only have the trivial solution $c_1 = c_2 = c_3 = 0$

Therefore, the polynomials are Linearly Independent

Example

In the vector space $V = P(\mathbb{R})$, consider the subset $S = \{x-1, x^2+1, x^3-x^2-x+3\}$. Is S linearly dependent or linearly independent?

Consider $$a_1(x-1)+a_2(x^2+1)+a_3(x^3-x^2-x+3)=0$$

Equating the coefficients of the powers of x to zero for each term in the LHS, we get -

$$\begin{aligned} -a_1 + a_2 + 3a_3 &= 0 \\ a_1 \qquad\quad - a_3 &= 0 \\ a_2 - a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$

The only solution to this linear homogenous system is the trivial solution, so the vectors in the subset S are **linearly independent**

BASIS OF A VECTOR SPACE (Definition)

Let $\mathbb{X}$ be a non-empty subset of a vector space $\mathcal{V}$. Then $\mathbb{X}$ is called a *basis* of $\mathcal{V}$ if **both** the following are true:

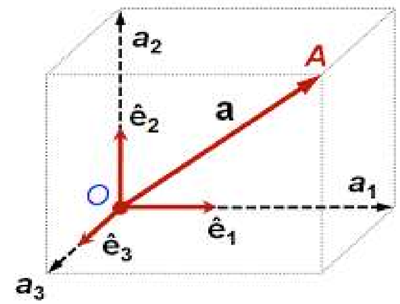
i. $\mathbb{X}$ is linearly independent — cannot generate an element of $\mathbb{X}$ as linear combination of the other elements of $\mathbb{X}$

ii. $\mathbb{X}$ generates $\mathcal{V}$ (i.e. $\mathbb{X}$ *spans* $\mathcal{V}$) — any element of $\mathcal{V}$ can be generated as a linear combination of the elements of $\mathbb{X}$

What is the meaning of "*spans*"?

Technically, it means that every element (vector) in the space $\mathcal{V}$ can be expressed as a linear combination of the elements of the set $\mathbb{X}$.

Examples of Bases



1. Basis of $\mathbb{R}^n$: $\boldsymbol{e_1} = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \ \boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ 0 \end{pmatrix}, \ \ldots, \ \boldsymbol{e_n} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^n$ because (i) they are

linearly independent (by inspection), and (ii) they *span* $\mathbb{R}^n$ because $c_1\boldsymbol{e_1} + c_2\boldsymbol{e_2} + \cdots c_n\boldsymbol{e_n} = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix}$

generates any vector in $\mathbb{R}^n$ depending on the values of $c_i \forall i = 1,2,\ldots,n$

Examples of Bases (continued):

2. Let $\mathbb{P}_n$ be a vector space of all polynomial functions of degree n or less. The basis of $\mathbb{P}_n$ is $\{1, x, x^2, \ldots., x^n\}$, the set of monomials.
(This is not a unique basis set because $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ also forms a basis where $p_i(x)$ is a polynomial in $\mathbb{P}_n$ of degree i.)

3. Let $\mathbb{M}_{m\times n}(\mathbb{F})$ denote the set of $m \times n$ matrices with entries in $\mathbb{F}$. Then $\mathbb{M}_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$. Vector addition is just matrix addition and scalar multiplication is defined in the obvious way (by multiplying each entry of the matrix by the same scalar). The zero vector is just the zero matrix. One possible choice of basis is the matrices with a single entry equal to 1 and all other entries 0.
(We will study the vector space of matrices in more detail in subsequent lectures!)

PROPERTIES OF BASES:

1. Must every vector space have a basis?

*Ans: Every **non-zero, finitely generated** vector space has a basis!*

2. Does a vector space have a unique basis?

Ans: Usually a vector space will have many bases. e.g., the vector space $\mathbb{R}^2$ *has the basis* $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\}$ *as well as the standard basis* $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$*.*

3. What is the dimension of a vector space?

Ans: $dim(\mathcal{V}) = no.\ of\ elements\ (vectors)\ in\ the\ basis\ (basis\ set)$.

Can you think of a vector space whose dimension is infinite?

A Few Other Things –

Finitely Generated Vector Space: One where you only need a finite number of elements to generate the vector space using linear combinations, e.g. $\mathbb{R}^2$ needs only (0,1) & (1,0) to generate all vectors in $\mathbb{R}^2$

Infinite Dimensional Vector Space (example): Let P be the vector space of all polynomials in X with rational coefficients. P is infinite dimensional. To see this – If P is given by the span of k polynomials in P, $p_1 \ldots. p_k$ where m is the maximum of the degrees of $p_1 \ldots. p_k$. Then x^{m+1} is a vector which cannot be written as a combination of $p_1 \ldots. p_k$. This is a contradiction so P cannot be finite dimensional.