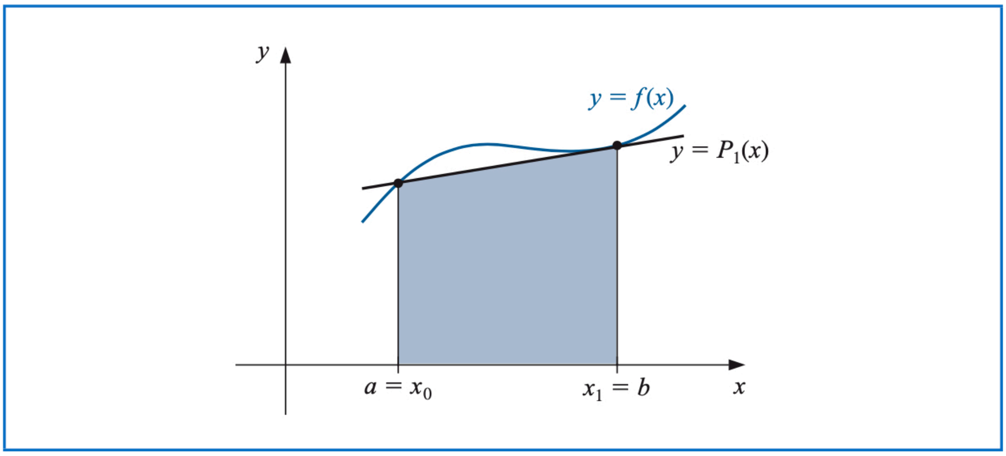
Numerical Integration

Objective: approximate $\int_a^b f(x)dx$ numerically by $\sum_{i=0}^{n} w_i f(x_i)$ where w_i are the appropriate weights, and x_i are the distinct nodes with $x_0 = a$ and $x_n = b$.

**** Several numerical integration schemes are available**

**** In this lecture, we will study the trapezoidal and Simpson's (1/3) rules**

Trapezoidal numerical integration:



** Here the basic idea is to use the weights $w_i = L_{1,i}(x)$ (linear Lagrange interpolating polynomial)

** We will derive the formula first geometrically, and then algebraically.

Geometrical derivation:

$\int_a^b f(x)dx$ = area under the curve $f(x)$ between $(x = a)$ and $(x = b)$

= area of the shaded trapezoid = $\dfrac{b-a}{2}\big(f(x_0) + f(x_1)\big)$

Algebraic derivation:

This derivation will automatically give us the error in the approximation along with the formula for the trapezoidal numerical integration!

Since we will use the linear Lagrange interpolating polynomials as our weight functions, the approximating polynomial for $f(x) \approx P_1(x) = \dfrac{x - x_1}{x_0 - x_1} f(x_0) + \dfrac{x - x_0}{x_1 - x_0} f(x_1)$

Error term of the Lagrange interpolation

So, $\displaystyle\int_a^b f(x)dx$

$$= \int_{x_0}^{x_1} \left(\frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \right) dx + \frac{1}{2} \int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1) dx$$

$$= \int_{x_0}^{x_1} \left(\frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \right) dx + \frac{1}{2} \boxed{f''(\xi) \int_{x_0}^{x_1} (x - x_0)(x - x_1) dx}$$

$$= \frac{x_1 - x_0}{2} \Big(f(x_0) + f(x_1) \Big) - \frac{h^3}{12} f''(\xi)$$

$$= \frac{h}{2} \Big(f(x_0) + f(x_1) \Big) - \frac{h^3}{12} f''(\xi);$$ **where** $\xi \in (x_0, x_1)$.

weighted mean value theorem for integrals

** for any function $f(x)$ for whose second derivative is identically 0! (eg. polynomial of degree one or less) uses second order Lagrange polynomial with equi-spaced nodes: x0, x1, x2

Simpson's Rule:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi).$$

The error term in Simpson's rule involves the fourth derivative of f, so it gives exact results when applied to any polynomial of degree three or less.

Question: Compare the Trapezoidal rule and Simpson's rule approximations to $\int_0^2 f(x)\,dx$ when $f(x)$ is

(a)	x^2	**(b)**	x^4	**(c)**	$(x+1)^{-1}$
(d)	$\sqrt{1+x^2}$	**(e)**	$\sin x$	**(f)**	e^x

Solution: On $[0, 2]$ the Trapezoidal and Simpson's rule have the forms

$$\text{Trapezoid:} \quad \int_0^2 f(x)\,dx \approx f(0) + f(2) \quad \text{and}$$

$$\text{Simpson's:} \quad \int_0^2 f(x)\,dx \approx \frac{1}{3}[f(0) + 4f(1) + f(2)].$$

When $f(x) = x^2$ they give

$$\text{Trapezoid:} \quad \int_0^2 f(x)\,dx \approx 0^2 + 2^2 = 4 \quad \text{and}$$

$$\text{Simpson's:} \quad \int_0^2 f(x)\,dx \approx \frac{1}{3}[(0^2) + 4 \cdot 1^2 + 2^2] = \frac{8}{3}.$$

The approximation from Simpson's rule is exact because its truncation error involves $f^{(4)}$, which is identically 0 when $f(x) = x^2$.

The results to three places for the functions are summarized in Table. In each instance Simpson's Rule is significantly superior.

Table

	(a)	(b)	(c)	(d)	(e)	(f)
$f(x)$	x^2	x^4	$(x+1)^{-1}$	$\sqrt{1+x^2}$	$\sin x$	e^x
Exact value	2.667	6.400	1.099	2.958	1.416	6.389
Trapezoidal	4.000	16.000	1.333	3.326	0.909	8.389
Simpson's	2.667	6.667	1.111	2.964	1.425	6.421