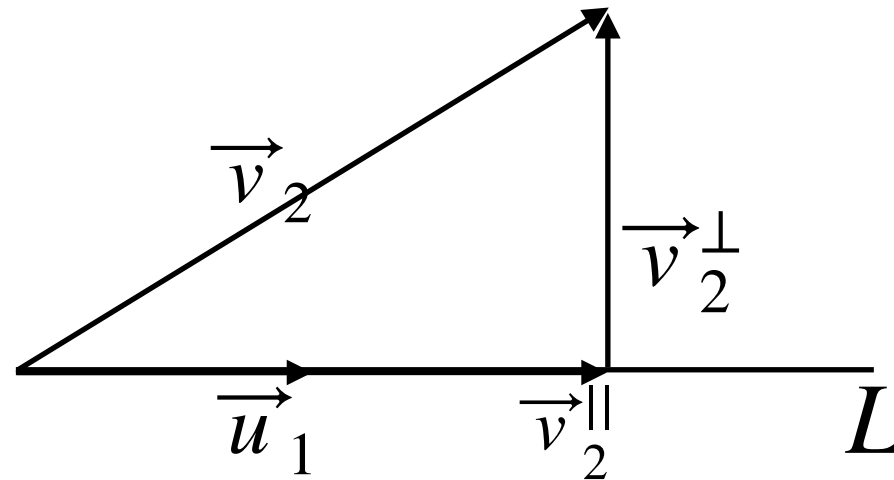


How to construct orthonormal basis vectors?

Let us illustrate this procedure by an example of a vector space V which is a plane. Given that V has a basis \vec{v}_1 and \vec{v}_2 which are **NOT** orthogonal, how do we construct an orthonormal basis?

Step 1: The first basis vector is easy to construct: $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1$.



Step 2: Construct $\vec{v}_2^{\parallel} = \text{proj}_L(\vec{v}_2) = \langle \vec{u}_1, \vec{v}_2 \rangle \vec{u}_1$

Step 3: Construct $\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel}$.

Step 4: Construct $\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|} \vec{v}_2^{\perp}$

Example: Find an orthonormal basis \vec{u}_1, \vec{u}_2 of the subspace $V = \text{span} \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 9 \\ 1 \end{pmatrix} \right)$ of \mathbb{R}^4 with basis

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 9 \\ 9 \\ 1 \end{pmatrix}.$$

Solution: Using the procedure mentioned above, we arrive at $\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$

Gram-Schmidt orthonormalisation process

Consider the basis $\vec{v}_1, \dots, \vec{v}_m$ of a subspace V of \mathbb{R}^n . For $j = 2, \dots, m$; we resolve the vector \vec{v}_j into its components parallel and perpendicular to the span of the preceding vectors $\vec{v}_1, \dots, \vec{v}_{j-1}$:

$$\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp}, \quad \text{with respect to } \text{span}(\vec{v}_1, \dots, \vec{v}_{j-1}).$$

$$\text{Then } \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1, \quad \vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|} \vec{v}_2^{\perp}, \dots, \vec{u}_j = \frac{1}{\|\vec{v}_j^{\perp}\|} \vec{v}_j^{\perp}, \dots, \vec{u}_m = \frac{1}{\|\vec{v}_m^{\perp}\|} \vec{v}_m^{\perp}$$

Is an orthonormal basis of V . Here $\vec{v}_j^{\perp} = \vec{v}_j - \vec{v}_j^{\parallel} = \vec{v}_j - \langle \vec{u}_1, \vec{v}_j \rangle \vec{u}_1 - \dots - \langle \vec{u}_{j-1}, \vec{v}_j \rangle \vec{u}_{j-1}$.