3.3 Total Differential

`(Snapshot)` (x_0, y_0) to $(x_0 + dx, y_0 + dy)$; the resulting differential in f is

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

This change in the linearization of f is called the **total differential** of f.

`Reading Assignment` Review examples 3,5 and 6 (page 938-940) from your textbook. Also, review section (12.5) on chain rule and its different forms from your textbook.

4 Directional Derivatives and Gradient Vectors

4.1 Directional Derivative

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\hat{u} = u_1\hat{i} + u_2\hat{j}$ is the *scalar* number:

$$\left(\frac{df}{ds}\right)_{\hat{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists. Note that arguments $x = x_0 + su_1$ and $y = y_0 + su_2$ is analogous to the parametric equation of a line through P_0 parallel to $\hat{u}$.
The directional derivative is a special case of the Gâteaux derivative in the context of locally convex topological vector spaces. Gâteaux differential is often used to formalize the functional derivative commonly used in the calculus of variations and physics. Unlike other forms of derivatives, the Gâteaux differential of a function may be nonlinear.

Notation: $(D_{\hat{u}}f)_{P_0} \implies$ derivative of f at P_0 in the direction of $\hat{u}$.

4.2 Gradient Vector or Gradient

Gradient of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)_{P_0}\hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\hat{j}$$

obtained by evaluating the partial derivatives of f at P_0.

Theorem: If the partial derivatives of $f(x, y)$ are defined at P_0; then

$$(D_{\hat{u}}f)_{P_0} = (\nabla f)_{P_0} \cdot \hat{u}$$

4.3 Properties of Directional Derivatives

$(D_{\hat{u}}f) = \nabla f \cdot \hat{u} = |\nabla f||\hat{u}|\cos\theta = |\nabla f|\cos\theta$

1. f increases most rapidly when $\cos\theta = 1$ or when $\hat{u}$ is in the direction of ∇f, i.e. f increases most rapidly in the direction of ∇f at any point P in the domain of f.
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$.

4.3.1 Sample review problem (example)

Let $f(x, y) = e^{x^2-y^2} \forall(x, y)$ in R^2. Note $f : R^2 \rightarrow R$ is continuously differentiable (why ?)
Clearly, $\frac{\partial f}{\partial x}(1, 1) = 2$, $f_y(1, 1) = -2$ and therefore, $\nabla f(1, 1) = 2\hat{i} - 2\hat{j}$. Hence, the direction in which the function f is increasing the fastest at $(1, 1)$ is given by the unit vector $\hat{v} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

So, remember

1. The direction of $\nabla f(x)$ is the direction of maximal increase of f at x.
2. The norm, $||\nabla f(x)||$ is equal to the magnitude of the rate of change of f in its direction of maximal increase.

4.4 Real world application: (a little more physics will do nobody any harm ☺)

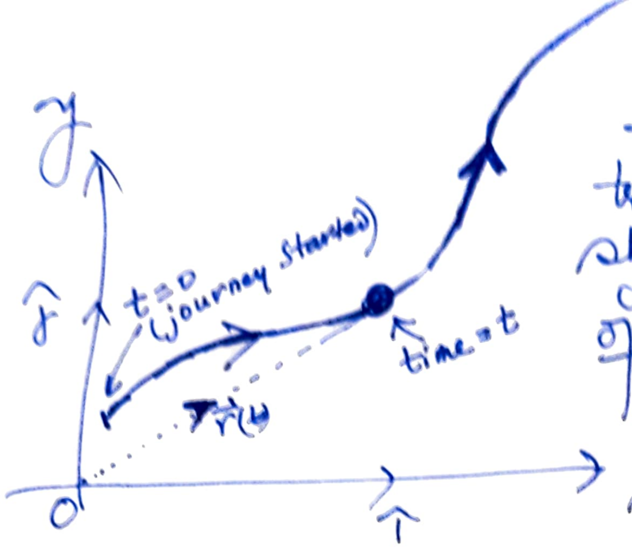
Law of Conservation of Energy: By now, "Sum of potential and kinetic energy is constant "should be more familiar to you than any kind of rock music ever composed by *The Beatles* ! ☺(`Oh ya ! The B's are one of my favorites`).
Lets prove this famous result from Physics using the vector calculus tools that we have just learnt ! But let's first begin with a few definitions.
Potential energy: If $\mathcal{F}$ is a vector **field** and if $\exists$ a differentiable function ϕ s.t. $\mathcal{F} = -\nabla\phi$; then ϕ is called the potential energy of the vector field, $\mathcal{F}$ and $\mathcal{F}$ is called *conservative* for the following reasons: Suppose that a particle of mass, m moves along a differentiable curve $\vec{r}(t)$ in U, where U is any open set in R^n; and it obeys *Newton's Laws*: $\mathcal{F}(\vec{r}(t)) = m\vec{r}''(t) \forall t$ where $\vec{r}(t)$ is defined. Recall: $\vec{F} = m\vec{a}$

Let us know formally state the *law of conservation of energy* as "If $\mathcal{F} = -grad\ \ \phi = -\nabla\phi$; then the sum of potential and kinetic energy is constant."
Proof: We have to prove: $\phi(\vec{r}(t)) + \frac{1}{2}m\vec{r}'(t)^2$ is constant; let us differentiate this sum and apply chain rule $\nabla\phi(\vec{r}(t)).\vec{r}'(t) + m\vec{r}'(t).\vec{r}''(t) = \nabla\phi(\vec{r}(t)).\vec{r}'(t) + \Big(-\nabla\phi(\vec{r}(t))\Big).\vec{r}'(t) = 0$; where the second last equality is obtained by $m\vec{r}'(t) = \mathcal{F}(\vec{r}(t)) = -\nabla\phi(\vec{r}(t))$. Hence proved ! □

5 Tangent Planes and Normal lines

If $f(x, y)$ is differentiable, and the level curve of f is given by $f(x, y) = c$; or equivalently if $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$ and $f(g(t), h(t)) = c$ (constant);
differentiating both sides we get; $\frac{d}{dt}f(g(t), h(t)) = \frac{d}{dt}(c) = 0 \implies \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$ (Chain rule)

Just think of g(t) as x(t) and h(t) as y(t)!



the blob is tracing a trajectory shown by the curve, w/ passage of time, and moving in a dirⁿ shown by the arrow.
∴ this trajectory $\vec{r}(t)$ is parametrized by the time variable t.

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$f(x(t), y(t))$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{d(c)}{dt} = 0$$

↑ Total derivative of f

$$\underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right)}_{\vec{r}\,'(t)} = 0$$

i.e $\nabla f \perp \vec{r}\,'(t)$

↑ this is tangent to the curve $\vec{r}(t)$.

$\Longrightarrow \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\left(\frac{dg}{dt}\hat{i} + \frac{dh}{dt}\hat{j}\right) = 0 \Longrightarrow \nabla f \perp \frac{d\vec{r}}{dt}$. i.e. $\nabla f \perp$ tangent to the curve $\Longrightarrow \nabla f \perp$ curve, $\vec{r}(t)$. The figure(2) shows that at every point (x_0, y_0) in the domain of f, $\nabla f \perp$ the level curve through (x_0, y_0).

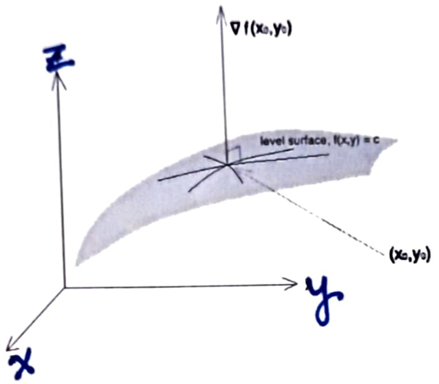


Figure 2: $\nabla f \perp$ level curve through (x_0, y_0), ©*Undergraduate Analysis, Serge lang*

Definitions:

1. The **Tangent Plane** at the point $P_0 = (x_0, y_0 z_0)$ on the level surface $f(x, y, z) = c$ is the plane through $P_0 \perp (\nabla f)_{P_0}$. Equivalently, the equation of the tangent plane to $F(x, y, z) = 0$ at $P_0(x_0, y_0, z_0)$ is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

2. The **Normal line** of the surface at P_0 is the line through P_0 parallel to $(\nabla f)_{P_0}$

5.1 Sample review problem

Find the equation of the tangent plane and the normal to the surface $z^2 = 4(1 + x^2 + y^2)$ at $(2, 2, 6)$
Soln. : Tangent plane: $4x + 4y - 3z + 2 = 0$ and Normal line $\frac{x-2}{4} = \frac{y-2}{4} = \frac{z-6}{-3}$

5.2 Increments and Distance

To estimate the change in the value of f when we move a small distance ds from a point P_0 in a particular direction $\hat{u}$; we use

$$df = \left((\nabla f)_{P_0} \cdot \hat{u}\right)(ds)$$

6 More on Directional Derivatives and Gradients

6.1 Why should the definition of Directional Derivative in §(4.1) in Handout 5 make sense?

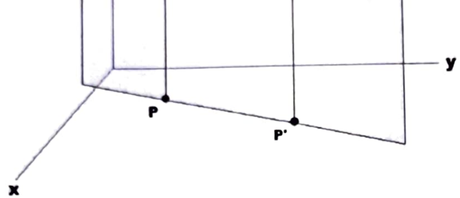


Figure 4: Directional derivative

6.2 Interpretation of the Directional derivative through an example

Let us consider an inverted right circular cone whose axis coincides with the z-axis,

$$z = f(x, y) = (x^2 + y^2)^{1/2} \tag{2}$$

We seek the directional derivative of this function at some point $x = a$ and $y = b$ and in the direction specified by $\hat{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$, figure(6).
The gradient of f is $\nabla f = f_x\hat{i} + f_y\hat{j} = \frac{x\hat{i}+y\hat{j}}{z}$ and therefore,

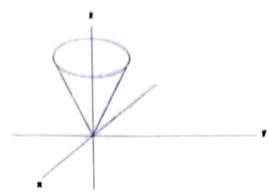


Figure 5: Right circular cone

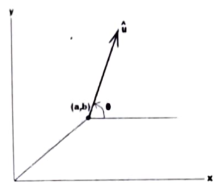


Figure 6: arbitrary direction, $\hat{u}$

$$\left(\frac{df}{ds}\right)_{x=a,y=b} = \hat{u} \cdot (\nabla f)_{x=a,y=b} = \frac{a\cos\theta + b\sin\theta}{\sqrt{a^2+b^2}} \tag{3}$$

Now, let us consider two cases.

1. Let θ be chosen such that $\hat{u}$ is in the radial direction in the xy-plane as shown in figure(9c) This means $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$ and so

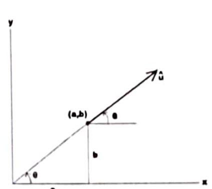


Figure 7: radial direction, $\hat{u}$

$$\frac{df}{ds} = \frac{a}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\frac{b}{\sqrt{a^2+b^2}} = 1$$

The significance of this result is shown in figure(8)

2. Now, we choose θ s.t. $\hat{u}$ is $\perp$ to the $\hat{u}$ in case(1), figure(9)

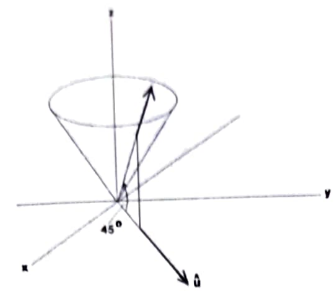


Figure 8: fastest rate of increase corresponding to radial $\hat{u}$

We then have $\cos\theta = \frac{-b}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$ and so

$$\frac{df}{ds} = \frac{a}{\sqrt{a^2+b^2}}\frac{-b}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}} = 0$$

The meaning of this result is illustrated in figure(9)

Figure 9: zero rate of increase corresponding to transverse û (*locked in circular loop, hence no increase in* $z = f(x,y)$)

Note that the direction of ∇f at (a,b) is obtained from $\nabla f = \frac{a\hat{i}+b\hat{j}}{\sqrt{a^2+b^2}}$ which is the same direction as $\hat{u}$ (case 1) in figure(8) and is also the direction of the fastest rate of change of f(x,y) at that point. However, if the direction of $\hat{u}$ was chosen as in case(2) i.e. $\perp$ to the radial direction; then the rate of change of f is 0, i.e. we cannot go up the surface of the cone but keep making circles at the same altitude, refer figure(9).

So as was stated before in section(4.3), *the gradient of a scalar function $F(x,y,z)$ is a vector that is in the direction in which F undergoes the greatest rate of increase and that has the magnitude equal to the rate of increase in that direction.*

Recall, the electric field at a point is equal to the negative gradient of the electric potential there, i.e. $\vec{E} = -\nabla\phi$. Why does this make sense ? Since, $\nabla\phi$ is a vector in the direction of *increasing* ϕ, the force on the *positive* charge q is $\vec{F} = q\vec{E} = -q\nabla\phi$, which is in the direction of *decreasing* ϕ. Thus, the negative sign ensures that a positive charge moves `downhill` from a higher to a lower potential.