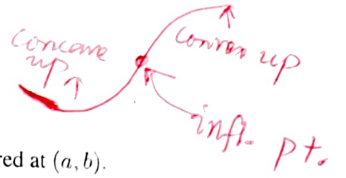
Date: $22^{nd} - 23^{rd}$ June, 2010

1 Extreme points and Saddle points

1.1 Local extrema

For a bi-variate function $f(x, y)$, we look for points where the surface $z = f(x, y)$ has a *horizontal tangent plane*; at such points we then look for local maxima, local minima and saddle points (think of inflection points in 1D).



1.1.1 Definition:

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** of f if $f(a, b) \geq f(x, y) \forall$ domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local miniimum** of f if $f(a, b) \leq f(x, y) \forall$ domain points (x, y) in an open disk centered at (a, b).

Thus, local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms. At such points, the tangent planes, if they exist, are horizontal.

1.1.2 Theorem: (First derivative test for local extrema)

If $f(x, y)$ has a local maximum or minimum at an interior point (a, b) of its domain, and if the first partial derivatives exist there; then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Warning: The theorem does not apply to

1. boundary points of a function's domain, where it is possible for a function to have extreme values along with non-zero derivatives.
2. points where either f_x or f_y fails to exist.

1.1.3 Definition:

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is called a **critical point** of f.

def^n of c.p.

Note:

1. All extreme values of f occur at critical points and/or boundary points.
2. Not every critical point gives rise to a local extremum. (may be a saddle point)

1.1.4 Definition:

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a **saddle point** of the surface.

1.1.5 More on Saddle point

In mathematics, a saddle point is a point in the domain of a function which is a stationary point but not a local extremum. The name derives from the fact that in two dimensions the surface resembles a saddle that curves up in one direction, and curves down in a different direction (like a horse saddle or a mountain pass). In terms of contour lines, a saddle point can be recognized, in general, by a contour that appears to intersect itself. For example, two hills separated by a high pass will show up a saddle point, at the top of the pass, like a figure-eight contour line.

In the most general terms, a saddle point for a smooth function (whose graph is a curve, surface or hypersurface) is a stationary point such that the curve/surface/etc. in the neighborhood of that point is not entirely on any side of the tangent space at that point. In one dimension, a saddle point is a point which is both a stationary point and a point of inflection. Since it is a point of inflection, it is not a local extremum.

In dynamical systems, a saddle point is a periodic point whose stable and unstable manifolds have a dimension which is not zero. If the dynamic is given by a differentiable map f then a point is hyperbolic if and only if the differential of n (where n is the period of the point) has no eigenvalue on the (complex) unit circle when computed at the point.

In a two-player zero sum game defined on a continuous space, the equilibrium point is a saddle point. A saddle point is an element of the matrix which is both the smallest element in its column and the largest element in its row. For a second-order linear autonomous systems, a critical point is a saddle point if the characteristic equation has one positive and one negative real eigenvalue.

Figure (1) shows a typical saddle point.

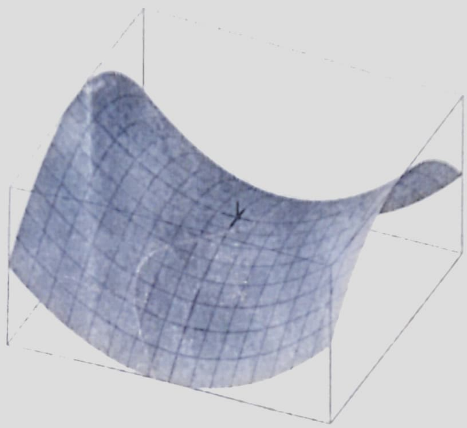


Figure 1: A saddle point on the graph of $z = x^2 - y^2$

1.1.6 Theorem: (second derivative test for local extrema)

Let $f(x, y)$ and its first and second partial derivatives be continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$; also define the **discriminant** of f as

$$\Im = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

then

1. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $\Im > 0$ at (a, b).
2. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $\Im > 0$ at (a, b).
3. f has a **saddle point** at (a, b) if $\Im < 0$ at (a, b).
4. Inconclusive at (a, b) if $\Im = 0$ at (a, b).

* Solve an example problem for local extremum of f(x,y).

1.2 Absolute maxima and minima on closed bounded regions

The following steps must be executed.

1. *List the interior critical points of* f in the region $R$and evaluate f at these points.
2. *List the boundary points of* R *where* f has local maxima and minima and evaluate f at these points.
3. Pick the absolute maxima and minima from the list of candidate points obtained in steps 1 and 2 above.

1.3 Sample review problem:

Ques: Near Kate's lake in Frodo Baggins National Park, the elevation of the solid ground (in feet) can be described by the function $f(x, y) = 8000 - 30x^2y^2 + 30x^2 + 30y^2$.

1. Determine the coordinate location, (x, y, z) corresponding to the bottom of the Kate's lake.
2. Determine the coordinate location(s), (x, y, z) corresponding to potential drainage points of Kate's lake.
3. What is the maximum possible depth of Kate's lake ? State clearly if information provided is insufficient to calculate it's depth.

Q) Find the critical pts. for each of the following f^n's and use the second derivative test to find the local extrema.

$z = f(x,y) = 4x^2 + 9y^2 + 8x - 36y + 24$

Ans)

Step ①:- C.P.

$f_x(x,y) = 8x + 8 = 0$

$f_y(x,y) = 18y - 36 = 0$

$\Rightarrow (-1, 2)$ is a c.p. of f

Step ② Discriminant of $f(x,y)$

$$J = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}_{(-1,2)}$$

$$= \begin{vmatrix} 8 & 0 \\ 0 & 18 \end{vmatrix}_{(-1,2)} = 144 > 0$$

$f_{xx}(-1,2) = 8 > 0$

$\Rightarrow$ C.P. is a local min. of $f(x,y)$

When we hit a local min/max f, if we move along our path (expedition), f(x,y) will seem like not to change (much) → This means we have changed upon a contour line of f(x,y); At this opportune epoch in our expedition, the contour lines of f(x,y) & g(x,y) = 0 coincide ⇒ ∇f & ∇g coincide i.e. $\nabla f = \lambda \nabla g$.

Soln:

1. Set $\nabla f = 0$ to obtain $(0,0)$ and $(\pm 1, \pm 1)$ as critical points. Then check $\Im(0,0) = 60^2 > 0$ and $f_{xx} = 60 > 0 \implies (0,0,8000)$ is the minimum and hence the location of the bottom of the lake !
2. $(\pm 1, \pm 1)$ are the **drainage (i.e. saddle) points** since $\Im(\pm 1, \pm 1) = -4(60^2) < 0$.
3. The maximum depth of the lake is $f(\pm 1 \ \pm 1) - f(0,0) = 8030 - 8000 = 30ft$.

Reading assignment: Review example 5, page 974 from the textbook.

2 Constrained optimization

2.1 Method of Lagrange multipliers

We wish to optimize (minimize or maximize) a function $f(x,y)$ subject to the constraint $g(x,y) = 0$.

Steps:

1. Set $\nabla f(x,y) = \lambda \nabla g(x,y)$. Q) Why should we solve this to find min/max of f?
2. Solve for λ.
3. Substitute λ in your constraint equation.
4. Solve for x and y.

Ans) the constraint $g(x,y) = 0$ means we can only ride along the level curve in our expedition to find pts. that min or max $f(x,y)$ #

Warning:

1. To find the minima and the maxima you must plug in all the candidate points in f and check thereafter which points give maxima and which points give minima. **You should not use any theorem or results from section 1 above to identify the minima from the maxima** as those results don't work with constraint optimization problems.
2. ! The condition $\nabla f = \lambda \nabla g$ is **not sufficient** but only a necessary condition of an extreme value of $f(x,y)$ subject to $g(x,y) = 0$. For example, try using this method to maximize $f(x,y) = x + y$ subject to $xy = 16$. The method will identify two points $(4,4)$ and $(-4,-4)$ as potential candidates for the extrema; yet the sum $(x+y)$ has no maximum value on the hyperbola $xy = 16$ because the farther one goes from the origin on this hyperbola in the first quadrant the larger the sum $(x+y)$ becomes !

$\nabla f = \hat{i} + \hat{j}$

$\nabla g = y\hat{i} + x\hat{j}$

$\nabla f = \lambda \nabla g \Rightarrow \hat{i} + \hat{j} = \lambda y \hat{i} + \lambda x \hat{j}$

Reading assignment: Review example 4, page 985 from textbook.

2.2 Sample exercise problems:

1. Find the points on the surface $z^2 = xy + 4$ closest to the origin.
2. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy plane that are nearest to and farthest from the origin.
3. The base of an open-top rectangular box costs \$3 per square meter to construct, the sides cost only \$ 1 per square meter. Find the dimension of the box of greatest volume that can be constructed for exactly \$ 36.

$\lambda y = 1$

$\lambda x = 1$

$\Rightarrow \frac{1}{y} = \frac{1}{x}$

$\Rightarrow \frac{x}{y} = 1$

$\Rightarrow x = y$

2.2.1 Lagrange multipliers with two constraints

To optimize $f(x,y,z)$ subject to $g_1(x,y,z) = 0$ and $g_2(x,y,z) = 0$, we introduce two Lagrange multipliers λ and μ; i.e. we locate the points where f takes on its constrained extreme values by finding x, y, z, λ and μ that simultaneously satisfy the following equations:

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x,y,z) = 0, \quad g_2(x,y,z) = 0$$

Reading assignment: Review example 5, page 986 from textbook.

$\lambda = \frac{1}{y} = \frac{1}{x}$

$xy = 16$

$(4,4)$ $x^2 = 16$

$x = \pm 4$

2.2.

Q1) $r^2 = x^2 + y^2 + z^2 = f(x, y, z)$

$z^2 - xy - 4 = 0 = g(x, y, z)$

Solve for λ!

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda \left\{ -y\hat{i} - x\hat{j} + 2z\hat{k} \right\}$$

Compare $\hat{i}, \hat{j}, \hat{k}$ components –

$2x = -\lambda y$; $2y = -\lambda x$; $2z = \lambda(2z)$

$\lambda = -\dfrac{2x}{y}$ $\quad \lambda = -\dfrac{2y}{x}$; $\lambda = 1$

$$-\frac{2x}{y} = -\frac{2y}{x}$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow y = \pm x$$

Now plug in $g(x, y, z) = 0$ → Case (2)

Case (1) $z^2 = x^2 + 4^2$ or $z^2 = -x^2 + 4$

Case ①

Pts. are $(x, +x, \sqrt{4+x^2})$
and $(x, +x, -\sqrt{4+x^2})$

Case (2) Pts are $(x, -x, \sqrt{4-x^2})$
and $(x, -x, -\sqrt{4-x^2})$

Plug

But recall $\lambda = 1$

$\Rightarrow -\frac{2x}{y} = 1 \Rightarrow y = -2x$ & $-2y = xz$

$\Rightarrow -2(-2x) = x$

$\Rightarrow 4x - x = 0$

$\Rightarrow x = 0$

Pts. are $(0,0,2)$ & $(0,0,-2)$

We need to min. $r^2 = x^2+y^2+z^2$

$\therefore$ Both pts min r^2.

2.2.2 Sample exercise problems:

1. Find the extreme value of $w = xyz$ on the line of intersection of the two planes $x + y + z = 40$ and $x + y - z = 0$. Give a geometric argument to support your claim that you have found a maximum and not a minimum value of w.

2. Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.

3. Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

The following section is only for motivational purpose

2.3 The foundations of the theory of Lagrange multipliers

A compendium on the development of the theory of Lagrange multipliers is provided at the following web link for those who crave for more !

`http://amath.colorado.edu/courses/2350/2010Sum/R025Notes/lagrange-multiplier.pdf`

max constraint

Q 2.2(2) $f(x,y) = x^2+y^2$; $g(x,y) = x^2+xy+y^2-1=0$

$\nabla f = \lambda \nabla g$ gives $\rightarrow \lambda = \dfrac{2x}{2x+y}$; $D^r \neq 0$

$\lambda = \dfrac{2y}{x+2y}$; $D^r \neq 0$

$\therefore \dfrac{2x}{2x+y} = \dfrac{2y}{x+2y}$ gives $y = x$, $y = -x$

Plug case (1) in $g = 0$ — Plug case (2) in $g = 0$

Candidate pts: $\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$, $\left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)$ — $(1,-1)$, $(-1,1)$

Compute $f(x,y)$ for each of the 4 points:

① $f(x,y) = 2/3$ for $\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$

② $f(x,y) = 2/3$ for $\left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)$

③ $f(x,y) = 2$ for $(+1,-1)$

④ $f = 2$ for $(-1,1)$

$\therefore$ the farthest pts are $(1,-1)$ & $(-1,1)$ & the farthest dist. is $\boxed{\sqrt2}$.

At pt. ⊗; $f(x,y) = d^*$ coincides w/ $g(x,y,z) = 0$